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## LETTER TO THE EDITOR

## Sharp crossover and anomalously large correlation length in driven systems

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Online at [stacks.iop.org/JPhysA/35/L459](http://stacks.iop.org/JPhysA/35/L459)**Abstract**

Models of one-dimensional driven diffusive systems sometimes exhibit an abrupt increase in the correlation length to an anomalously large but finite value as the parameters of the model are varied. This behaviour may be misinterpreted as a genuine phase transition. A simple mechanism for this sharp increase is presented. The mechanism is introduced within the framework of a recently suggested correspondence between driven diffusive systems and zero-range processes. It is shown that when the dynamics of the model is such that small domains are suppressed in the steady-state distribution, anomalously large correlation lengths may build up. The mechanism is examined in detail in two models.

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Driven diffusive systems have been studied extensively in recent years [1]. Particular attention has been given to one-dimensional models, which have been shown to behave very differently from systems in thermal equilibrium [2, 3]. For example, in contrast to equilibrium systems with short-range interactions, non-equilibrium systems, characterized by dynamics which do not obey detailed balance, may exhibit spontaneous symmetry breaking and phase separation even when the dynamics are local.

It has recently been noted that numerical simulations may be rather misleading in attempting to determine whether or not a particular model exhibits phase separation. For example, direct numerical simulations of a single-lane three-species model introduced by Arndt *et al* [4] (AHR) strongly indicate that two phase transitions take place in the model. The first transition is from a homogeneous state to a mixed state, in which the system separates into two phases: one with a high density and the other with a low density. In the second transition, which takes place from the latter state, the system further phase separates into three pure phases. Analytical studies of the model [5] have shown that the mixed state suggested by the numerical studies is in fact homogenous, and the model exhibits only one transition, from the

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homogeneous to the three-phase state. At the apparent transition to what was misinterpreted as a mixed state, the model exhibits a sharp increase of its correlation length  $\xi$  to an anomalously large but finite value. The change in  $\xi$  is very sharp and it occurs over a narrow range of parameters defining the model. Such a behaviour may be easily misinterpreted in numerical studies of systems smaller than  $\xi$  as a genuine phase transition.

Recently a simple criterion for the occurrence of phase separation in one-dimensional systems has been proposed [6]. This criterion does not rely on direct numerical simulations, and it enables one to determine whether or not a particular model exhibits phase separation. It has been applied to a two-lane driven model, introduced by Korniss *et al* [7]. Direct numerical simulations of this model have indicated that it exhibits phase separation in a region of its parameter space. Moreover, numerical simulations, to be discussed below, show what seems like a sharp transition between a homogeneous and a phase separated state as the parameters defining the model are varied. On the other hand, the criterion conjectured in [6] indicates that in fact this model, too, does not exhibit phase separation. Thus the apparent phase separation observed in the numerical studies may again be due to a sharp crossover which results in a large but finite correlation length. The criterion [6] addresses the issue of the existence of phase separation; however, it has not provided an insight into why the correlation length becomes anomalously large in certain cases, and why the increase in  $\xi$  takes place over such a narrow range of model parameters.

In this letter, we suggest a simple mechanism which accounts for the large correlation lengths and sharp crossover phenomena observed in some driven systems. We show that when the dynamics are such that small domains of the high-density phase are suppressed in the steady-state distribution, the resulting correlation length becomes anomalously large, leading to an apparent phase separation in numerical studies of finite systems. The mechanism is discussed within the framework put forth in [6]. In this framework a correspondence is made between one-dimensional driven systems and zero-range processes. We use this correspondence to analyse the crossover phenomena observed in the AHR and the two-lane models, and discuss the underlying mechanism in detail.

We start by briefly reviewing the physical framework of the criterion introduced in [6]. In this framework the dynamics of the driven system are modelled by a zero-range process (ZRP). Such a process [8, 9] is defined on a one-dimensional lattice of  $M$  sites, or ‘boxes’, with periodic boundary conditions. Particles, or ‘balls’, are distributed among the boxes, with the box  $i$  occupied by  $n_i$  balls. At each time step a box  $i$  is chosen at random and a ball is removed from it and transferred to one of its nearest neighbours with a rate  $w_{n_i}$ . The rate  $w_{n_i}$  depends only on the occupation number,  $n_i$ , in that box. The steady-state weights of the ZRP are known to have the form [8, 9]

$$W_{\text{ZRP}}(\{n_i\}) = \prod_{i=1}^M z^{n_i} \mathcal{F}_{n_i} \quad (1)$$

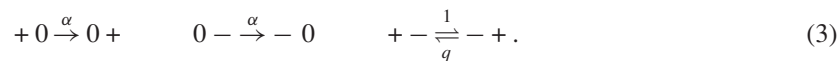
where  $z$  is the fugacity,  $\mathcal{F}_k = \prod_{m=1}^k 1/w_m$  for  $k \geq 1$ , and  $\mathcal{F}_0 = 1$ . The correspondence to driven systems is made by identifying occupied boxes with domains of the high-density phase. Specifically, a configuration of the driven system can be described as a sequence of high-density domains separated by low-density intervals. In the ZRP, each high-density domain is represented by a box, and the number of balls in that box corresponds to the domain length. The evolution of the driven model may be studied by considering the currents flowing in and out of the high-density domains. The dynamics may be represented by a ZRP, with the rate  $w_n$  taken as the current  $J_n$  leaving a domain of size  $n$ . Within this picture, the existence of phase separation in the driven model corresponds to a macroscopic occupation of one of the boxes in the ZRP.

In [6] it was conjectured, based on the correspondence to a ZRP, that the existence of phase separation is related to the asymptotic large  $n$  behaviour of the current  $J_n$  on blocks of size  $n$ . For  $J_n$  of the form

$$J_n = w_n = J_\infty \left( 1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right) \right) \quad (2)$$

with  $J_\infty > 0$ , the existence of phase separation depends only on the coefficient  $b$ . Phase separation takes place at high densities only if  $b > 2$  and does not take place at any density if  $b < 2$ . Moreover, when  $J_n$  is asymptotically decreasing to zero at large  $n$ , phase separation takes place at any density.

We now utilize the correspondence between the ZRP and driven systems to gain insight into the sharp crossover phenomena discussed above in the AHR and two-lane models. We begin by considering the AHR model. This is a three-state model on a ring. Each site is either empty (0) or occupied by a positive (+) or a negative (−) particle. The model evolves by random sequential dynamics in which a pair of nearest neighbour sites are chosen at random and exchanged with the following rates:



The dynamics conserve the densities of each type of particle. The two-particle densities are taken to be equal. Exact calculations within the grand-canonical ensemble [5] have shown that this model has two states: a fully ordered state for  $q > 1$ , in which the system strongly phase separates into three phases (+, − and 0), and a disordered state for  $q < 1$  where particles and vacancies are homogeneously distributed. In the regime  $q < 1$  the current is an analytical function of  $q$ . However, in the vicinity of a particular value  $q_0 < 1$  (which depends on  $\alpha$  and on the density  $\rho$ ), a sharp crossover takes place from a regime in which the correlation length  $\xi$  is relatively small ( $q < q_0$ ) to a regime in which  $\xi$  is anomalously large, but finite ( $q_0 < q < 1$ ). In this regime the correlation length was found to reach values of the order  $10^{70}$  for some range of parameters. This makes it clear why numerical simulations of chains of length of the order of a few thousand suggest a phase-separated state for  $q_0 < q < 1$ .

To make the correspondence of the AHR model to the ZRP, high-density domains are identified as uninterrupted sequences of positive and negative particles bounded by vacancies. The number of boxes  $M$  in the ZRP is equal to the number of vacancies in the AHR model, and the number of balls  $N$  is equal to the number of positive and negative particles. Hence the mean occupation of a box in the ZRP,  $\phi = N/M$ , is related to the density  $\rho$  in the AHR by  $\rho = \phi/(1 + \phi)$ . The dynamics within a domain are given by the rates for the positive and negative particle exchange in (3). The flow of particles in and out of a domain is controlled by the exchange rate  $\alpha$  of particles with vacancies.

The dynamics within each domain are thus given by the well-studied partially asymmetric exclusion process (PASEP) [10, 11] with particles injected and ejected from the boundaries with rate  $\alpha$ . For this process the asymptotic form of the current is known to take the form (2) with  $J_\infty = (1 - q)/4$ . The coefficient  $b$  of the leading order correction is  $3/2$  for  $q < 1 - 2\alpha$  and  $-1$  for  $q > 1 - 2\alpha$ . Note that in the case  $b = -1$  the current is an increasing function of the domain size. Therefore small domains are more stable than large ones, and large correlation lengths cannot build up. We therefore concentrate only on the regime  $b = 3/2$ , where the sharp crossover has been observed.

It has been noted that the domain size distribution of the AHR model is exactly given by that of the corresponding ZRP [6]. Taking  $w_n$  to order  $1/n$  it is straightforward to show that

for large  $n$  the domain size distribution is given by

$$P(n) \sim \frac{1}{n^b} \exp(-n/\xi) \quad \xi = \frac{1}{|\ln(z/J_\infty)|} \quad (4)$$

with  $b = 3/2$ . The correlation length  $\xi$  is determined by the equation for the density

$$\phi = \frac{\sum_{n=0}^{\infty} n P(n)}{\sum_{n=0}^{\infty} P(n)}. \quad (5)$$

Since  $b < 2$  a finite correlation length  $\xi$  may be found for any density  $\phi$ . Thus, the distribution (4) is valid for any density and no phase separation takes place in this case. On the other hand for  $b > 2$  the distribution (4) can support only sufficiently low densities; and therefore one expects phase separation at high densities with a macroscopic occupation of one of the boxes, in analogy to Bose–Einstein condensation. To this order,  $\xi$  is determined by the density and is independent of  $q$ . Therefore the  $q$  dependence of  $\xi$  and the sharp crossover observed in simulations must come from higher-order corrections to the current. As will be shown below, the main effect of these corrections is to suppress the small domains. This results in a large correlation length in the system.

To demonstrate this point we first consider  $J_n$  to order  $1/n^2$ . We therefore study the ZRP with the current  $J_n$  given by

$$J_n = J_\infty \left( 1 + \frac{b}{n} + \frac{c}{n^2} \right). \quad (6)$$

In making the correspondence to the AHR model we take  $b = 3/2$ , and note that to leading order in  $(1 - q)$  the current  $J_n$  takes the form [12]

$$J_n = \frac{1 + (1 - q)f(n, \alpha)}{n - 1 + 2/\alpha} \quad (7)$$

where  $f(n, \alpha)$  is independent of  $q$ . Expanding this expression in powers of  $1/n$  we find that to leading order in  $(1 - q)$ , the coefficient  $c$  diverges as  $(1 - q)^{-1}$ . Thus,  $c$  (and in fact all higher-order corrections) is  $q$ -dependent, and becomes large near  $q = 1$ . Using the form (6) of the current, the domain size distribution for large  $n$  (4) becomes

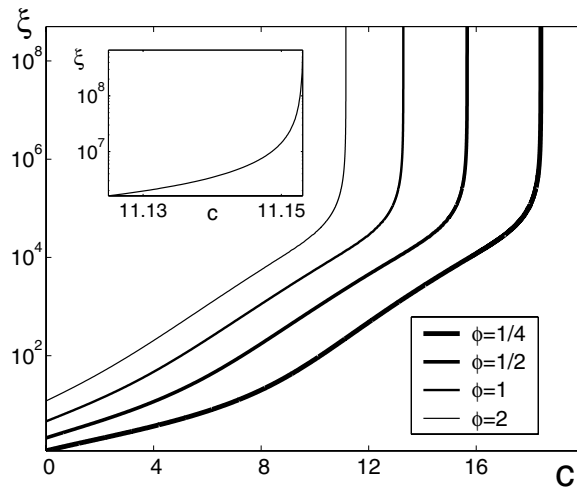
$$P(n) \sim \frac{1}{n^b} \exp(-n/\xi - c/n). \quad (8)$$

The value of  $\xi$  for a given density is determined by (5), and is therefore a function of  $c$ . In figure 1 we plot this function for several values of the density  $\phi$ . For any density the correlation length  $\xi$  exhibits two distinct regions in  $c$  with a sharp crossover between them but no singularity. For small  $c$  the correlation length seems to increase exponentially with  $c$ , while for large  $c$  the increase appears to be faster than exponential. This can be seen in the inset of figure 1, where we zoom in on a very narrow range of values of  $c$  at the crossover. The crossover point increases with  $\phi$ . Since, as argued before,  $c$  of the AHR model diverges at  $q = 1$ , we expect the crossover value to take place for some  $q < 1$ , as observed in direct numerical simulations of the model and in accordance with the exact results.

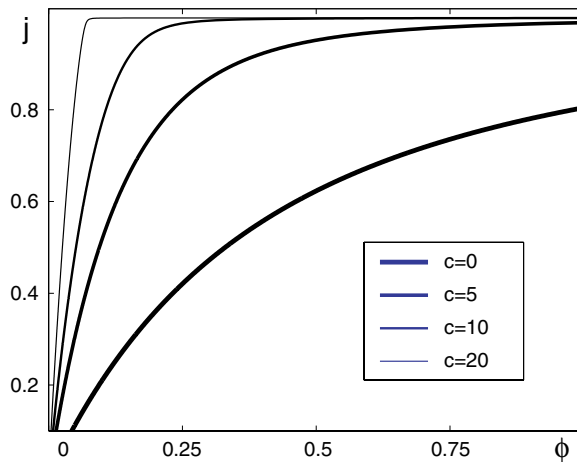
We now note that a similar sharp crossover phenomenon also takes place when one considers the behaviour of the current as a function of the density. Within the ZRP the quantity which corresponds to the current of the AHR model is

$$j = \frac{\sum_n n w_n P(n)}{\sum_n (n + 1) P(n)} \quad (9)$$

where  $w_n$  is given by the current  $J_n$  of (6) to order  $1/n^2$  with  $J_\infty = (1 - q)/4$ . It is straightforward to show that in the grand-canonical ensemble  $j = z$ . Using this relation,



**Figure 1.** The correlation length  $\xi$  as a function of the coefficient  $c$  for the ZRP with the transition rates given by (6). In the inset we zoom on the large  $\xi$  region for the case  $\phi = 2$ , showing that in this region  $\xi$  grows faster than exponentially. Note that  $\xi$  is plotted on a logarithmic scale in both graphs.



**Figure 2.** The current  $j$  as a function of  $\phi$  for the same model as in figure 1. For simplicity here we set  $J_\infty = 1$ .

we plot  $j$  as function of  $\phi$  for fixed values of  $c$  in figure 2. Here again two distinct regions can be observed for sufficiently large  $c$ . At low density  $j$  exhibits a notable increase with  $\phi$ , while it slowly approaches its maximal value,  $j = 1$ , at high densities. Between the two regimes one finds a sharp crossover but no singularity. The current density relation obtained here for the ZRP has the same features as that of the AHR model (compare figure 2 of [5] to figure 2 here).

To get a better understanding of this behaviour, we note that the effect of large  $c$  on the domain size distribution is to suppress the weight of small domains. This results in a large correlation length needed to sustain the density. In the AHR model, where the correspondence to the ZRP is exact, this suppression is achieved by large high-order corrections to the

current  $J_n$ . From figure 1 it can be seen that for a given density the sharp crossover occurs only for sufficiently large  $c$ . In this case it is evident that the high-order corrections can take high enough values as  $q$  is increased, since these corrections diverge at  $q = 1$ . However, in other models this suppression may be induced by other mechanisms, which do not require large high-order corrections to the current of the form discussed above. To demonstrate this point we consider a ZRP defined by the transition rates

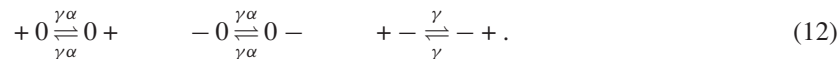
$$w_n = \begin{cases} W & n < n_0 \\ J_\infty(1 + b/n) & n \geq n_0 \end{cases} \quad (10)$$

where  $W$  and  $n_0$  are free parameters, and  $W$  directly controls the weight of small domains. Sufficiently large  $W$  destabilizes small domains and suppresses them in the steady-state distribution. This model with  $b = 0$  has been discussed previously in the context of ZRP and demonstrated to exhibit a sharp crossover by Evans [9]. A straightforward calculation leads to the domain size distribution

$$P(n) \sim \begin{cases} e^{-n/\xi_1} & n < n_0 \\ (n/n_0)^{-b} e^{-n_0/\xi_1} e^{-(n-n_0)/\xi_2} & n > n_0 \end{cases} \quad (11)$$

with  $\xi_1^{-1} = -\ln(z/W)$  and  $\xi_2^{-1} = -\ln(z/J_\infty)$ . Note that the  $n > n_0$  expression is valid only for large  $n$ . In order to suppress small domains one needs  $J_\infty < W$  which yields  $\xi_1 < \xi_2$ . Larger  $n_0$  or  $W$  result in a stronger suppression of small domains, increasing the weight of the tail of the distribution. We have calculated  $\xi = |\ln(z)|^{-1}$  as a function of  $W$  for a fixed density, and found that it exhibits similar features to those of figure 1, with a sharp crossover taking place at a value of  $W$  which decreases with  $n_0$ . For example, for  $n_0 = 5$ , the crossover takes place at  $W \simeq 8$ . We thus conclude that the sharp crossover is a direct result of suppression of small domains. The suppression of small domains results in an increase of the tail of the domain size distribution in order to keep the average density of particles. This results in a larger correlation length. Of course, the details of the suppression mechanism may differ from model to model and will affect the exact form of the distribution.

We now turn to consider the two-lane model introduced in [7]. The model is defined on a  $2 \times L$  periodic lattice, where each site is either empty or occupied by a positive or negative particle. The dynamics within a lane are governed by the rates (3) with  $q = 0$ . Particles are also allowed to hop between lanes with the rates

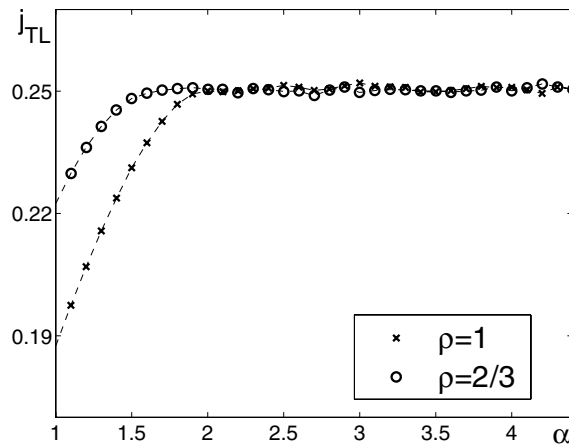


Direct numerical simulations suggest that for large  $\alpha$  the model exhibits phase separation [7].

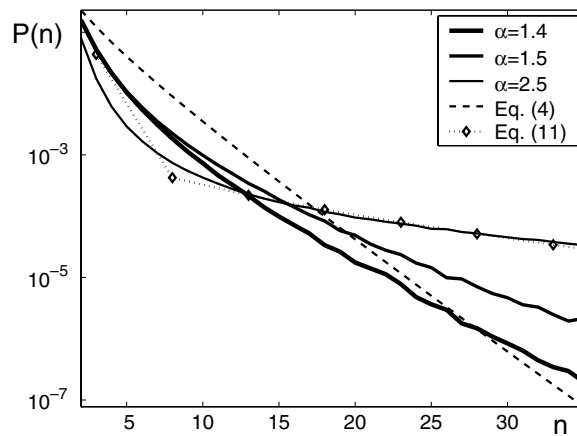
This model has recently been studied [6] using its conjectured correspondence to the ZRP, where the existence of a phase transition is determined by the leading finite-size correction to the current. It was suggested that the model does not exhibit phase separation. In analogy with the analysis of the AHR model, the current of a finite domain has been studied by considering an open system of length  $n$ . This system is fully occupied by positive and negative particles. In the interior the dynamics follows the same rates of the model given by equations (3) and (12), although here there are no vacancies. At the boundaries, positive (negative) particles are injected with rate  $\alpha$  at the left (right). This automatically implies that particles are removed at the other end with the same rate  $\alpha$ . The dynamics within a domain are therefore given by the particle exchange rates (3) and (12) with  $q = 0$  in the interior, together with the following rates at the boundaries of both lanes



This is a generalization of the totally asymmetric exclusion process with open boundary conditions to two lanes. For this model no analytical expression for the current  $J_n$  of a domain



**Figure 3.** The current  $j_{TL}$  of positive particles in the two-lane model, as a function of  $\alpha$ , for two densities.  $j_{TL}$  was obtained through direct numerical simulations of systems of size 5000.



**Figure 4.** The domain size distribution in the two-lane model for several values of the model parameter  $\alpha$ . Here  $\rho = 1/2$  and  $\gamma = 1$ . A domain of size  $n$  is defined as an uninterrupted sequence of length  $n$  of particles residing in both lanes. The normalization is such that domains of size 0 are not counted. Note that as  $\alpha$  increases small domains are more suppressed, and the tail of the distribution becomes more pronounced. For comparison we plot the distribution of the ZRP where small domains are not suppressed (dashed line), and the one where small domains are suppressed as given by equation (11) (dotted line, here we arbitrarily take  $W = 4$  and  $n_0 = 6$ ).

of size  $n$  is available. Extensive numerical studies [6] have shown that the asymptotic form of the current is given by equation (2) with  $b \simeq 0.8$  for any  $\gamma \neq 0$ . Therefore the model does not phase separate at any value of the parameters. However, as in the AHR model, keeping the density fixed one observes in numerical simulations of finite systems a sharp crossover in the current as the parameter  $\alpha$  is changed. This can be seen, for example, in figure 3, where the current  $j_{TL}$  of a two-lane system is plotted as a function of  $\alpha$ . Here, too, the sharp crossover seen in simulations may be misinterpreted as a phase transition.

We now show that in the regime where phase separation seems to take place, the statistical weight of small domains is suppressed, in accordance with the mechanism suggested above.



To this end we have studied the domain size distribution in the two-lane model by direct numerical simulations (see figure 4). It is instructive to compare these results with the distribution function (4) with  $b = 0.8$ , obtained from the ZRP where small domains are not destabilized. One can see that small domains in the two-lane system are significantly suppressed as compared with this distribution. This suppression is easily understood since small domains which form on one lane can readily dissolve through the exchange to the other lane. As discussed above this could lead to a sharp crossover in the correlation length. For comparison we also plot the distribution function (11). Although one does not expect the details of the distribution to agree it is evident that the general behaviour is captured.

In summary, a simple mechanism for the occurrence of a sharp crossover to anomalously large correlation length exhibited in some driven systems is suggested. The mechanism is examined in detail for two models, and is shown to be consistent with analytical results in one model and with numerical results in the other.

### Acknowledgments

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